Mathematics 131B, Fall 2016 – ℝami Luisto
Addon 01 - sup-metric is a metric.

We went through this proof on Friday lecture September 23rd. The proof is a bit technical, and the setting is quite abstract (the points of our space are functions) but this is a very important example of a metric for us. By studying metrics instead of explicitly defined distance functions we pay a cost of abstractness, but we gain a lot since we can apply the same machinery to a plethora of different scenarios. If we only studied Euclidean spaces the cost would be quite steep w.r.t. the benefits, but especially in the domain of functions spaces (e.g. $C([0,1],\mathbb{R})$) we can get a lot of use of the theory of metric spaces after we go through the effort of making sure we actually have a metric! Getting our hands dirty with the technicalities at this point will save us from a lot of sweat and tears later on.

Definition 1. A pair (X,d), where X is a set and $d\colon X \times X \to \mathbb{R}$ a mapping, is a *metric space* (and d a *metric*) if the following conditions hold:
a) $d(x,x) = 0$ for all $x \in X$,
b) for $x, y \in X$ with $x \neq y$, we have $d(x,y) > 0$,
c) $d(x,y) = d(y,x)$ for all $x, y \in X$, and
d) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Definition 2. Let $C([0,1],\mathbb{R})$ be the collection of all continuous mappings $f\colon [0,1] \to \mathbb{R}$. The mapping

$$d_\infty\colon C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R}) \to \mathbb{R}, \quad d_\infty(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$$

is called *the* sup-*metric*.

Proposition 3. *The* sup-*metric is a metric, i.e.* $(C([0,1],\mathbb{R}), d_\infty)$ *is a metric space.*

Proof. We go through the definitions of a metric one by one. In the following, f, g and h are continuous functions $[0,1] \to \mathbb{R}$.
a) By a direct calculation[1] we see that

$$d_\infty(f,f) = \sup_{x\in[0,1]} |f(x) - f(x)| = \sup_{x\in[0,1]} 0 = 0.$$

b) We want to show that if $f \neq g$, then $d_\infty(f,g) > 0$. We prove this by showing that if $d_\infty(f,g) = 0$, then $f = g$.[2] We see that

$$d_\infty(f,g) = 0 \Leftrightarrow \sup_{x\in[0,1]} |f(x) - g(x)| = 0.$$

By the definition of supremum, we thus have for all $y \in [0,1]$ that

$$|f(y) - g(y)| \leq \sup_{x\in[0,1]} |f(x) - g(x)| = 0.$$

[1]If this is difficult to follow, I recommend reviewing the defintion (and use) of the supremum.
[2]Why are these two claims equivalent?

Since for all $y \in [0,1]$, $|f(y) - g(y)| \geq 0$, we see that in fact $|f(y) - g(y)| = 0$ for all $y \in [0,1]$. This means that $f(y) = g(y)$ for all $y \in [0,1]$, so $f = g$ and the condition b) holds true.

c) By a direct calculation we see that[3]

$$\begin{aligned} d_\infty(f,g) &= \sup_{x\in[0,1]} |f(x) - g(x)| \\ &= \sup_{x\in[0,1]} |(-1)\cdot(g(x) - f(x))| \\ &= \sup_{x\in[0,1]} |(-1)|\cdot|(g(x) - f(x))| \\ &= \sup_{x\in[0,1]} 1\cdot|(g(x) - f(x))| \\ &= \sup_{x\in[0,1]} |(g(x) - f(x))| \\ &= d_\infty(g,f) \end{aligned}$$

d) Note first that for all $y \in [0,1]$,

$$|f(y) - g(y)| = |(f(y) - h(y)) + (h(y) + g(y))| \leq |f(y) - h(y)| + |h(y) + g(y)|$$

since the absolute value is a norm. Thus we can deduce[4] that

$$\begin{aligned} d_\infty(f,g) &= \sup_{x\in[0,1]} |f(x) - g(x)| \\ &\leq \sup_{x\in[0,1]} \left(|f(x) - h(x)| + |h(x) + g(x)|\right). \end{aligned}$$

Since the functions f, h and g are continuous, so is the mapping

$$y \mapsto \left(|f(x) - h(x)| + |h(x) - g(x)|\right).$$

As continuous mapping defined on a closed interval, it attains its maximum value at some point $x_0 \in [0,1]$. Thus

$$\sup_{x\in[0,1]} \left(|f(x) - h(x)| + |h(x) - g(x)|\right) = |f(x_0) - h(x_0)| + |h(x_0) - g(x_0)|.$$

By the definition of a supremum, $|f(x_0) - h(x_0)| \leq \sup_{x\in[0,1]} |f(x) - h(x)|$, (and similarly for the other part) so we get

$$|f(x_0) - h(x_0)| + |h(x_0) - g(x_0)| \leq \left(\sup_{x\in[0,1]} |f(x) - h(x)|\right) + \left(\sup_{x\in[0,1]} |h(x) - g(x)|\right)$$

[3] If this is difficult to follow, I recommend reviewing the defintion (and use) of the supremum.

[4] Againg, if this feels hard, I recommend reviewing the basics of the supremum.

Combining all the inequalities, we have

$$\begin{aligned} d_\infty(f,g) &\le \sup_{x\in[0,1]} \left(|f(x)-h(x)| + |h(x)+g(x)|\right) \\ &\le \left(\sup_{x\in[0,1]} |f(x)-h(x)|\right) + \left(\sup_{x\in[0,1]} |h(x)+g(x)|\right) \\ &= d_\infty(f,h) + d_\infty(h,g). \end{aligned}$$

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