Mathematics 131B, Fall 2016 – ℝami Luisto
Addon 02 - Two definitions of openness.

We defined openness on the lecture of Wednesday, September 28th. There are a few ways of defining openness, each with good properties. The definition in the lectures and in the book, saying that a set is open if it contains none of its boundary points (and closed if it countains all of them) is especially when learning that a set can be open or closed, but 'often' a set is neither since it will contain a part of its boundary.

Another definition requires that for all points there exists a ball around them contained in the set. This definition is much more useful in certain proofs, and heuristically conveys that an open set is one where for each point, all the points 'near enough' will be also contained in the set. (In a way, in an open set you can always 'move around' a bit without 'falling off' the set.)

Now we show that these two definitions equal, so that we can use them interchangeably.

As a reminder, the boundary of a set A in a metric space (X, d) can be defined as

$$\partial A = \{x \in X \mid \text{ for all } r > 0, B_d(x,r) \cap A \neq \emptyset \neq B_d(x,r) \cap \complement A\}$$

Definition 1 (Book's definition of openness)*.* Let (X, d) be a metric space and $A \subset X$ a set. We say that A is *open*, if it contains none of its boundary points, i.e. $A \cap \partial A = \emptyset$.

Definition 2 (Alternate definition of openness)*.* Let (X, d) be a metric space and $A \subset X$ a set. We say that A is *open*, if for all points $x \in A$ there exists a radius $r > 0$ such that $B_d(x, r) \subset A$.

Theorem. *The two definitions above are equivalent.*

Proof. Suppose $A \cap \partial A = \emptyset$ and fix a point $x \in A$. Since $x \in A$ and $x \in B_d(x, r)$ for all $r > 0$, we have $A \cap B_d(x, r) \neq \emptyset$ for all $r > 0$. If $\complement A \cap B_d(x, r) \neq \emptyset$ for all $r > 0$, we would then have $x \in \partial A$, which is impossible by our assumption. Thus there exists a radius $r_0 > 0$ for which $\complement A \cap B_d(x, r_0) = \emptyset$. But this implies $B_d(x, r_0) \subset A$ and so the set A is open by the alternate definition.

Suppose next that for every point $x \in A$ there exists a radius $r_x > 0$ such that $B_d(x, r_x) \subset A$. By the definition of the boundary this means that $x \notin \partial A$, so none of the points in A are boundary points. This means just that $A \cap \partial A = \emptyset$, so the set is open w.r.t. the book's definition. □