Mathematics 131B, Fall 2016 – ℝami Luisto
Homework 6 - Due date Friday November 4th.

You are welcomed, even encouraged to discuss the homework problems together. If and when you do you should, however, acknowledge your collaboration and mention who you have been working with in your returned solutions. I sincerely hope that it goes without saying that copying someone else's work without thought is not 'working together', and while not technically disallowed, is very counterproductive for your learning process. While in certain previous courses one might get a good grade just my memorizing a few solution algorithms, many problems in this course require some level of understanding what is going on mathematically. This will be the rule rather than the exception in all further courses, and trying to mimic proofs without thinking will lead to misery after the first few weeks. Understanding what is happening, on the other hand, will lead to feelings of **great beauty** and **elegance**.

If you are having problems with an exercise, you can and should ask for help. However, if you glance at a problem and immediately shout "I don't know how to solve this!", you are not having problems with the exercise but rather you are having problems with not expending any effort to solve the problem. **"You never call any question impossible, said Harry, until you have taken an actual clock and thought about it for five minutes, by the motion of the minute hand. Not five minutes metaphorically, five minutes by a physical clock."** (*Harry Potter in Harry Potter and the Methods of Rationality.*) `\end{rant}`

Prepping problems. These problems will not be graded. They are voluntary, very basic problems that can help you get up to speed.

p1) Show that a compactly supported continuous function attains its maximum and minimum.
p2) Show that a compactly supported continuous function is uniformly continuous.
p3) Show that the convolution of two compactly supported functions is compactly supported.

Homework problems

(1) Show that the convolution of two continuous compactly supported functions is continuous.

The next three problems together give rise to the following result.

Theorem. *For any $\varepsilon > 0$ and $\delta \in (0,1)$ there exists an (ε, δ)-approximation to the identity which is a polynomial on $[-1,1]$ (i.e. the restriction of the map to the interval $[-1,1]$ is a polynomial).*

(2) Prove that for any real number $0 \leq y \leq 1$ and any natural number $n \geq 0$, that $(1-y)^n \geq 1 - ny$. (Hint: induction on n. Alternatively, differentiate w.r.t. y.)

(3) Show that $\int_{-1}^{1}(1-x^2)^n \,\mathrm{d}\,x \geq \frac{1}{\sqrt{n}}$. (Hint: for $|x| \leq 1/\sqrt{n}$, use the previous problem; for $|x| \geq 1/\sqrt{n}$ use the fact that $(1-x^2) \geq 0$.)

(4) Prove the theorem above; that there exists an approximation to identity which is a polynomial. I recommend to draw a rough picture of the polynomial in question as well. (Hint: set $f\colon \mathbb{R} \to \mathbb{R}$ to be $f(x) = c(1-x^2)^N$ for $x \in [-1,1]$ and $f(x) = 0$ for $x \notin [-1,1]$, where N is number sufficiently large and c chosen such that $\int_{-1}^{1} f = 1$. Use then the previous problem.)

The following is an extra problem to those who want to learn something more. It will not be graded.

Extra challenge: Let $f' \in C(\mathbb{R},\mathbb{R})$ and

$$\phi \in C_0^1(\mathbb{R},\mathbb{R}) = C^1(\mathbb{R},\mathbb{R}) \cap C_0(\mathbb{R},\mathbb{R}).$$

Show that $\phi' \in C_0(\mathbb{R},\mathbb{R})$ and that

$$\int_{-\infty}^{\infty} f'(x)\phi(x)\,\mathrm{d}\,x = \int_{-\infty}^{\infty} f(x)\phi'(x)\,\mathrm{d}\,x. \tag{D}$$

(Hint: Integration by parts.)

Using the integral identity (D) we may define the *distributional derivative* of a mapping $f \in C(\mathbb{R},\mathbb{R})$ to be any mapping g for which

$$\int_{-\infty}^{\infty} g(x)\phi(x)\,\mathrm{d}\,x = \int_{-\infty}^{\infty} f(x)\phi'(x)\,\mathrm{d}\,x$$

holds for any $\phi \in C_0^1(\mathbb{R},\mathbb{R})$. (If such a mapping g exists. A distributional derivative is also called a weak differential in this setting, and those mapping that have weak differentials are called weakly differentiable.)

Denote

$$H\colon \mathbb{R} \to \mathbb{R}, \quad H = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Show that for any $\phi \in C_0^1(\mathbb{R},\mathbb{R})$, $\int_{-\infty}^{\infty} \phi'(x)H(x)\,\mathrm{d}\,x = \phi(0)$, i.e. the notorious Dirac delta-function is the distributional derivative of H. (In whatever sense it may exist.)

(Hint: study the integral disjointly on $(-\infty,0)$ and $(0,\infty)$.

You might wish to try and plot the derivatives of the mapping you constructed in problems (2)-(4) and see if it corresponds to H. Note that since we are studying (weak) differentials of H, it does not matter if we study H or $H + \text{const}$.