Mathematics 131B, Fall 2016 – ℝami Luisto
Homework 7 - Due date Friday November 18th.

You are welcomed, even encouraged to discuss the homework problems together. If and when you do you should, however, acknowledge your collaboration and mention who you have been working with in your returned solutions. I sincerely hope that it goes without saying that copying someone else's work without thought is not 'working together', and while not technically disallowed, is very counterproductive for your learning process. While in certain previous courses one might get a good grade just my memorizing a few solution algorithms, many problems in this course require some level of understanding what is going on mathematically. This will be the rule rather than the exception in all further courses, and trying to mimic proofs without thinking will lead to misery after the first few weeks. Understanding what is happening, on the other hand, will lead to feelings of **great beauty** and **elegance**.

If you are having problems with an exercise, you can and should ask for help. However, if you glance at a problem and immediately shout "I don't know how to solve this!", you are not having problems with the exercise but rather you are having problems with not expending any effort to solve the problem. **"You never call any question impossible, said Harry, until you have taken an actual clock and thought about it for five minutes, by the motion of the minute hand. Not five minutes metaphorically, five minutes by a physical clock."** (*Harry Potter in Harry Potter and the Methods of Rationality.*) `\end{rant}`

Prepping problems. These problems will not be graded. They are voluntary, very basic problems that can help you get up to speed.

p1) Calculate the radius of convergence for $\sum \frac{1}{n!}x^n$.
p2) Find the Taylor series around 0 of the function $f(x) = 2x^2 - 5x + 1$.
p3)

Homework problems

(1) (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$. Find the Taylor series of f around $x_0 = 0$
(b) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the polynomial $f(x) = 3x^2 - 5x^2 + 2$. Find the Taylor series of f around $x_0 = 1$.

(2) We define, as in the book,
$$\exp\colon \mathbb{R} \to \mathbb{R}, \quad \exp(x) = \sum_{n=0}^{\infty} \frac{x^k}{k!}.$$
Show that $\exp'(x) = \exp(x)$. Use this to show that exp does not have any local extrema.

(3) Show that for every integer $n \geq 3$, we have

$$0 < \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots < \frac{1}{n!}.$$

(Hint: Show first that $(n+k)! > 2^k n!$ for all $k = 1, 2, 3 \ldots$.)

Use this to deduce that $n! \exp(1)$ is not an integer for any $n \geq 3$ and conclude that $\exp(1)$ is an irrational number.

(4) Let $f \colon \mathbb{R} \to \mathbb{R}$ be a real analytic function satisfying $f' = f$ and $f(0) = 1$. Show that $f = \exp$.

(Hint: the coefficients in the local power series expression of a real analytic function are unique.)

(5) Find the Taylor series of $f(x) = \cos(2x)$ around the origin. Do not use the formal power series definition of cos but instead calculate the derivatives of the function f using what you have learned in calculus.

What is the radius of convergence?

The following is an extra problem to those who want to learn something more. It will not be graded.

Extra challenge: Let $f \colon \mathbb{R} \to \mathbb{R}$ be a smooth function. Then we can write down the following **formal power series**:

$$T_f(x) = f(0) \sum_{n=1} \frac{f^{(n)}(0)}{n!} x^n.$$

This is called the *Taylor series* of f (around 0). From what we studied in class, we see that f is real analytic at 0 exactly when T_f converges in some interval $(-R, R)$ **and** $T_f(x) = f(x)$ for $x \in (-R, R)$.

Either of these two conditions can fail.

(a) Let us study the series $\sum_{n=0}^{\infty} \mathrm{e}^{-n} \cos(n^2 x)$. Show via the Weierstrass M-test that this series converges to a continuous mapping $g \colon \mathbb{R} \to \mathbb{R}$.
Note that since the mappings $f_n(x) = \mathrm{e}^{-n} \cos(n^2 x)$ are smooth, so is g. Show next that for k odd, we have $g^{(k)}(0) = \sum_{n=0}^{\infty} n^{2k} \mathrm{e}^{-n}$.
Furthermore show that

$$\frac{g^{(k)}(0)}{k!} \geq n^k \cdot \mathrm{e}^{-k}.$$

Deduce that the Taylor series T_g of g around 0 has radius of convergence 0. Thus g is a smooth map with a Taylor series at 0 not converging outside $x = 0$.

(b) Let now $f(x) = \exp(-\frac{1}{x^2})$. Convince yourself that $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$ and deduce that the Taylor series T_f of f around 0 converges, but $T_f(x) \neq f(x)$ for $x \neq 0$.

How can we figure out if a smooth function is real analytic?

We can write any smooth function $f\colon (-, R, R) \to \mathbb{R}$ in the form

$$f(x) = f(0) \sum_{n=1}^{k} \frac{f^{(n)}(0)}{n!} x^n + R_k(x),$$

where $R_k(x)$ is the so called *error term* of the Taylor series. From this we see that the map f is real analytic at 0 if and only if there is some interval $(-r, r) \subset (-R, R)$ such that $\lim_{k\to\infty} R_k(x) = 0$ for all $x \in (-r, r)$.

Note that for the mapping $f(x) = \exp(-\frac{1}{x^2})$ we actually have $R_k(x) = f(x)$ for all x, and as we noted earlier this map is not real analytic.

It is not hard to show that the error term can be written in the form

$$R_k(x) = \int_0^x \frac{(t-0)^k}{k!} f^{(k+1)}(t)dt.$$

By using this integral form, show that if there exists a constant C and $\rho > 0$ such that $|f^{(k)}(0)| \le C\frac{k!}{\rho^k}$ for all $k \in \mathbb{N}$, then $\lim_{k\to\infty} R_k(x) = 0$ for all $x \in (-\rho, \rho)$. Conclude that f is then real analytic at 0.

If you wish you may continue to show that if there exists a constant M such that $|f^{(k)}(0)| \le M$ for all $k \in \mathbb{N}$, then f is real analytic at 0. Using this you can show that a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $f' = f$ is not only smooth but actually real analytic. (With related methods one can show that a mapping satisfying $f' = 1/f$ is also real analytic.)