Mathematics 131B, Fall 2016 – $\mathbb{R}$ami Luisto
Homework 9 - Due date Friday December 2nd.

You are welcomed, even encouraged to discuss the homework problems together. If and when you do you should, however, acknowledge your collaboration and mention who you have been working with in your returned solutions. I sincerely hope that it goes without saying that copying someone else's work without thought is not 'working together', and while not technically disallowed, is very counterproductive for your learning process. While in certain previous courses one might get a good grade just my memorizing a few solution algorithms, many problems in this course require some level of understanding what is going on mathematically. This will be the rule rather than the exception in all further courses, and trying to mimic proofs without thinking will lead to misery after the first few weeks. Understanding what is happening, on the other hand, will lead to feelings of **great beauty** and **elegance**.

If you are having problems with an exercise, you can and should ask for help. However, if you glance at a problem and immediately shout "I don't know how to solve this!", you are not having problems with the exercise but rather you are having problems with not expending any effort to solve the problem. **"You never call any question impossible, said Harry, until you have taken an actual clock and thought about it for five minutes, by the motion of the minute hand. Not five minutes metaphorically, five minutes by a physical clock."** (*Harry Potter in Harry Potter and the Methods of Rationality.*) `\end{rant}`

Prepping problems. These problems will not be graded. They are voluntary, very basic problems that can help you get up to speed.

p1) Let $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$. Show that

$$\int_0^1 f(x)dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)dx$$

(Hint: $f(x-1) = f(x)$.)

Homework problems

(1) Let $f\colon \mathbb{R} \to \mathbb{C}$ be a function with $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$.
   (a) Suppose $f(-x) = f(x)$. Show that

$$\hat{f}(n) = \int_0^1 f(x)\cos(2\pi n x)dx.$$

   (Hint: Use prepping problem 2.)
   (b) Suppose $f(-x) = -f(x)$. Show that

$$\hat{f}(n) = \int_0^1 f(x)\sin(2\pi n x)dx.$$

   (Hint: Use prepping problem 2.)

(2) Let $f\colon \mathbb{R} \to \mathbb{C}$ be a function with $f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k)e_k(x)$.

(a) Suppose $\hat{f}(-n) = \hat{f}(n)$. Show that

$$f(x) = \hat{f}(0) + \sum_{k=1}^{\infty} \hat{f}(k)\cos(2\pi kx).$$

(b) Suppose $\hat{f}(-n) = -\hat{f}(n)$. Show that

$$f(x) = i\sum_{k=1}^{\infty} \hat{f}(k)\sin(2\pi kx).$$

(3) Define

$$g\colon \left[-\frac{1}{2}, \frac{1}{2}\right] \to \mathbb{R}, \quad g(x) = \frac{1}{2} - |x|$$

and extend it to a 1-periodic function $g\colon \mathbb{R} \to \mathbb{R}$ by setting $f(x+n) = g(x)$ for all $n \in \mathbb{Z}$.

(a) What is the Fourier series of f?

(Hint: Use problem 1 and show first that the k:th Fourier coefficient is just $2\int_0^{\frac{1}{2}} f(x)\cos(2\pi kx)dx$. *Then integrate by parts.)*

(b) Use the Fourier series above to calculate the exact value of the series[1]

$$\sum_{k=1,3,5,\ldots} \frac{1}{k^2}.$$

Here you may assume that the Fourier series of f converges to f pointwise.

(c) Apply the Plancherel formula to calculate the exact value of the series

$$\sum_{k=1,3,5,\ldots} \frac{1}{k^4}.$$

[1]Calculating the exact value of the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is known as the *Basel problem*. We will be content to solve this partial problem for now.

The following is an extra problem to those who want to learn something more. It will not be graded.

Extra challenge: (a) Show that if a function $f\colon \mathbb{R} \to \mathbb{R}$ is L-periodic for all $L > 0$, then it is a constant map.

(b) Let (a_n) be a sequence of real numbers such that $a_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = 0$.
Show that if $f\colon \mathbb{R} \to \mathbb{R}$ is a_n-periodic for all $n \in \mathbb{N}$, then it is a constant map.

(c) Let $L_1, L_2 > 0$ be real numbers such that $L_1 \neq qL_2$ for any rational number $q \in \mathbb{Q}$.
Show that if $f\colon \mathbb{R} \to \mathbb{R}$ is both L_1-periodic and L_2-periodic, then it is a constant map.