Mathematics 131B, Fall 2016 – ℝami Luisto
Practice problems.

These are problems you can use to practice for the first midterm. (They will not be graded in any way.) Old homework problems are also good practice! If you really want to understand the subject (and have a lot of time) work through all the problems in Tao's book until the end of the Section 2.3. Continuity and Compactness. If your time is on a budget, I found the following problems to be especially useful in addition to the problems listed below:

- 1.1.3. (Constructing functions that fail exactly one of the conditions on a metric.)
- 1.1.16 (The distance of elements in converging sequences converge to the distance of their limits.)
- 1.2.2. (Proving equivalent conditions of the closure.)
- 1.2.3. (Proving the big list (a)-(h) of properties of open and closed sets.)
- 1.3.1. (Prove the theorem giving an equivalent definition of relative closedness.)
- Pretty much any problems in the section 1.4. defining completeness that ask you to prove some proposition are good practice.
- 1.5.7. (Proving the last theorem of the section telling the connection of compactness of closedness, compactness and finite unions and compactness of finite subsets.)
- 1.5.12. (Asking when the discrete metric space is compact or complete.)
- 2.1.1. and 2.1.2 (Proving two theorems characterizing continuity.)
- Pretty much any problems in the section 2.3.. discussing compactness and continuity that ask you to prove some proposition are good practice.

Metrics

M1. Equip $\mathbb{R}^2$ with the maximum metric

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

Show that the set $\{(x, y) \in \mathbb{R}^2 \mid x > 0\}$ is open in $(\mathbb{R}^2, d_\infty)$.

M2. Define in $\mathbb{R}^n$ the maximum metric

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

the euclidean metric

$$d_e(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2, \ldots, (x_n - y_n)^2}$$

and the taxicab metric

$$d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|.$$

Show that for any points $\mathbf{x}$ and $\mathbf{y}$:

$$d_e(\mathbf{x},\mathbf{y}) \le \sqrt{n} d_\infty(\mathbf{x},\mathbf{y}) \le \sqrt{n} d_1(\mathbf{x},\mathbf{y}) \le n\sqrt{n} d_e(\mathbf{x},\mathbf{y}).$$

M3. Show that for any normed vector space $(V, \|\cdot\|)$, the map $d_{\|\cdot\|}(v,w) = \|v-w\|$ is a metric.

M4. Give an example of a mapping $d\colon X \times X \to \mathbb{R}$ that is **not** a metric. (You may choose X to be what you wish.)

M5. Show that

$$d\colon X \times X \to \mathbb{R}, \quad d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y \end{cases}$$

is a metric for any (non-empty) set X.

Opennes, closure and boundary

O1. Let (X,d) be a metric space. Show that:
- (a) the open ball $B(x,r)$ is an open set for any $x \in X$, $r > 0$.
- (b) the set $X \setminus \{x_0, \dots, x_k\}$ is an open set for any finite collection of points $x_0, \dots, x_k \in X$.
- (c) if $A, B \subset X$ are open, then so are the sets $A \cup B$ and $A \cap B$.
- (d) the finite intersections and arbitrary unions of open sets of X are open.

O2. Suppose U is a non-empty open set in the plane and that $x_0 \notin U$. Can $U \cup \{x_0\}$ be open?

O3. (The methods needed for this problem will be studied on Wednesday.) Verify that the following sets are open. Hint: Make use of the fact that if f is continuous and U is open, then $f^{-1}(U)$ is open.
- $\{(x,y) \in \mathbb{R}^2 : x^2 - y^2 > 0\}$.
- $\{(x,y) \in \mathbb{R}^2 : 0 < x^2 - y < 1 \text{ and } x > y\}$.
- $\{(x,y) \in \mathbb{R}^2 : x^2 > y \text{ or } y^2 > x\}$.

O4. Let (X,d) be a metric space, $A \subset X$.
- Show that $\operatorname{int}(A)$ is the union of all open sets contained in A.
- Show that $\overline{A}$ is the intersection of all closed sets containing A.

O5. Let (X,d) be a metric space, $A \subset X$. We set the *diameter* of A to be $d(A) \in [0,\infty]$ to be

$$d(A) = \sup\{d(a,b) \mid a, b \in A\}.$$

- (a) Show that for the diameter of a ball of radius r we have $d(B(x,r)) \le 2r$. Give an example of a metric space and a ball where the inequality is strict.
- (b) Show that $d(A) < \infty$ if and only if A is bounded.[1]
- (c) Show that for a compact subset $K \subset X$ there exists points $a_0, b_0 \in K$ such that $d(K) = d(a_0, b_0)$.
- (d) Show that a sequence (x_n) is Cauchy if and only if for $S_m = \{x_n \mid n \ge m\}$ we have $\lim_m d(S_m) = 0$.

O6. Let (X,d) be a metric space, $A \subset X$. Define $d(x,A) = \inf_{a \in A} d(x,a)$.
- Prove that if A is closed and $x \notin A$, then $d(x,A) > 0$.

[1]A set if bounded if it is contained in a ball.

- Prove that $|d(x,A) - d(y,A)| \leq d(x,y)$.
- Prove that $\{x : d(x,A) = 0\} = \overline{A}$.

O7. Let $C([0,1],\mathbb{R})$ the space of continuous functions $f\colon [0,1] \to \mathbb{R}$ equipped with the supremum metric:

$$d_\infty(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

Take a continuous function $g\colon [0,1] \to \mathbb{R}$ and set

$$U = \{f \in C([0,1],\mathbb{R}) \mid f(x) > g(x) \text{ for all } x \in [0,1]\}.$$

Is U open in $(C([0,1],\mathbb{R}), d_\infty)$? What is the boundary ∂U?

O8. Let $C([0,1],\mathbb{R})$ the space of continuous functions $f\colon [0,1] \to \mathbb{R}$ equipped with the L_1-metric:

$$d_1(f,g) = \int_0^1 |f(x) - g(x)| \,\mathrm{d}\,x$$

(You may assume d_1 is a metric in $C([0,1],\mathbb{R})$.)

Let

$$U = \{f \in C([0,1],\mathbb{R}) \mid f(x) \geq 0 \text{ for all } x \in [0,1]\}.$$

Is U closed in $(C([0,1],\mathbb{R}), d_1)$? What is the boundary ∂U?

Completeness and compactness

CC1. Suppose that A and B are disjoint compact subsets of a metric space X. Show that "A and B are separated by a positive distance," in the sense that $\inf\{d(a,b) : a \in A, b \in B\} > 0$.

CC2. Let $A \subset \mathbb{R}$ be given by $A = \{0\} \cup \{1/n : n \in \mathbb{N}\}$. Prove that A is compact directly from the definition using open coverings. Prove that A is compact directly from the definition using sequences.

CC3. Let A be a closed subset of a metric space (X,d). Prove that:
- If (X,d) is complete, then (A,d_A) is complete.
- If (X,d) is compact, then (A,d_A) is compact.

CC4. Let (X,d) be a compact metric space. Show that X has a countable dense subset A by the following steps:
- For each n, show that X can be covered by finitely many balls of radius $1/n$.
- Let A_n be the set of center points of the balls of radius n from part (a). Let $A = \bigcup_{n=1}^\infty A_n$. Prove that $\overline{A} = X$.
- Verify that A is countable.

CC5. Let X be a metric space. Consider a nested sequence of nonempty compact sets $K_1 \supset K_2 \supset \dots$ Prove that $\bigcap_{n=1}^\infty K_n$ is nonempty and compact.

CC6. Suppose that X is a metric space. Let $C \subset X$ be closed and $K \subset X$ be compact. Show that $C \cap K$ is compact.

CC7. Let $X = (0,1]$. Define two different metrics $d(x,y) = |x - y|$ and $\rho(x,y) = |1/x - 1/y|$.

- Show that U is open with respect to d if and only if it is open with respect to ρ.
- Show that (X, ρ) is complete, but (X, d) is not complete.

Continuity

Cont1. Show that $|d(x,z) - d(y,z)| \leq d(x,y)$. Conclude that $d(x,z)$ is a continuous function of x.

Cont2. Let $X = C([0,1])$ with the metric $d(f,g) = \|f-g\|_\infty = \max_x |f(x) - g(x)|$.
- Define $I : X \to \mathbb{R}$ by $I(f) = \int_0^1 f(t)\, dt$. Show that I is continuous.
- Suppose that $F : \mathbb{R} \to \mathbb{R}$ satisfies $|F(x) - F(y)| \leq C|x-y|$ for some constant $C > 0$. Define $F_* : X \to X$ by $F_*(f) = F \circ f$ for $f \in X$. Show that F_* is continuous.

Cont3. *Urysohn's Lemma:* Suppose A and B are two disjoint closed subsets of a metric space X. Construct a continuous function $f : X \to [0,1]$ such that $f|_A = 0$ and $f|_B = 1$. [Hint: Use problem 6.]

Cont4. *Extreme Value Theorem:* Let X be a metric space, $K \subset X$ compact, $f : X \to \mathbb{R}$ continuous. Show that f attains a maximum and minimum on K.

Cont5. If A and B are disjoint, A is closed, and B is compact, prove that $\inf\{d(a,b) : a \in A, b \in B\}$ is still strictly positive. Hint: Apply the extreme value theorem to the function $f : B \to \mathbb{R}$ given by $f(x) = d(x, A)$.

More elaborate problems.

Fun1. Let (X,d) be a metric space $x_0 \in X$ and $a \in X$. We define a function $f_a \colon X \to \mathbb{R}$ by setting $f_a(x) = d(x,a) - d(x,x_0)$.

(a) Show that f_a is a bounded function. (I.e. that there exists a number $M \geq 0$ such that $-M \leq f_a(x) \leq M$ for all $x \in X$.)

(b) By (a) $f_a \in F(X,\mathbb{R})$ is bounded, so $f_a \in Bd(X,\mathbb{R})$. This means that we can define a mapping $\phi \colon X \to Bd(X,\mathbb{R})$ by setting $\phi(a) = f_a$. (Think about this for a moment.)

Let us equip $Bd(X,\mathbb{R})$ with the sup-metric:

$$d_\infty(f,g) = \sup_{x \in X} |f(x) - g(x)|.$$

The mapping ϕ is now a mapping between two metric spaces. Show that it is 1-Lipschitz, i.e.

$$d_\infty(\phi(a), \phi(b)) \leq d_X(a,b).$$

(c) Show that the mapping is even an *isometry*, i.e.

$$d_\infty(\phi(a), \phi(b)) = d_X(a,b).$$

(Hint: calculate the value of the function $\phi(a) - \phi(b)$ at a and b.)

Deduce the *Kuratowski embedding theorem*: every metric space can be isometrically embedded into some subset of a normed space. (Note that $Bd(X,\mathbb{R})$ is a normed space.)