Intro to Exchanging Integrals and Limits

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Notation: For a set S, we define the **indicator function of** S as

$$\chi_S(x) = \begin{cases} 1, & \text{if } x \in S \\ 0, & \text{if } x \notin S. \end{cases}$$

Problem 1. For each of the following sequences f_n, draw a picture of the graph of f_n and then compute

$$\lim_{n \to \infty} \int_0^\infty f_n(x)\,dx \text{ and } \int_0^\infty \left(\lim_{n \to \infty} f_n(x) \right) dx.$$

a. Spreading bump: $f_n = (1/n)\chi_{[0,n]}$.

b. Concentrating bump: $f_n = n\chi_{(0,1/n)}$.

c. Sliding bump: $f_n = \chi_{[n-1,n)}$.

d. Horizontally receding infinity: $f_n = \chi_{[n,+\infty)}$.

e. Vertically receding infinity: $f_n = 1/n$.

These are standard examples, but the specific names came from Terence Tao.

Problem 2. For $f : [a,b] \to \mathbb{R}$ continuous, denote $\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$. Recall that the Extreme Value Theorem guarantees that the supremum is achieved at some $x_0 \in [a,b]$ and thus $\|f\|_\infty$ is finite. Remember or look up the definition of uniform convergence. Then prove that $f_n \to f$ uniformly on $[a,b]$ if and only if $\lim_{n \to \infty} \|f_n - f\|_\infty = 0$.

Problem 3. The examples in Problem 1 show that you can't always exchange limits and integrals. However, suppose that $f_n : [a,b] \to \mathbb{R}$ is continuous and $f_n \to f$ uniformly. Prove that $\int_a^b f_n \to \int_a^b f$.

Problem 4. Define $f_n : [-\pi, \pi] \to \mathbb{R}$ by $f_n(x) = (1/n) \sin nx$.

1. Prove that $f_n \to 0$ uniformly.

2. Does $f_n' \to 0$?

3. What does this example illustrate?

Problem 5. Suppose that $f_n, f : [a, b] \to \mathbb{R}$. Assume that $f_n(0) \to f(0)$ and that $f_n' \to f'$ uniformly as $n \to \infty$.

a. Prove that $f_n(x) \to f(x)$ for every $x \in [a, b]$. **Hint:** Apply the fundamental theorem of calculus and Problem 3.

b. In fact, prove that $f_n(x) \to f(x)$ uniformly.

c. Conclude from (a) that $(d/dx)\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}(d/dx)f_n(x)$.

Challenge 1. Suppose $f_n : [a, b] \to \mathbb{R}$ is continuous and $f_n \to f$ uniformly. Suppose that $\{x_n\}$ is a sequence in $[a, b]$ and $x_n \to x_0$. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x_0).$$

Challenge 2. Suppose that X is a compact metric space. Assume that $f_n : X \to \mathbb{R}$ is continuous, and that $f_{n+1} \leq f_n$. Suppose that $f_n \to 0$ pointwise. Prove that $f_n \to 0$ uniformly.

Challenge 3. Let $f_n : [a, b] \to \mathbb{R}$. Suppose that f_n is Riemann-integrable and $f_n \to f$ uniformly. Prove that f is Riemann-integrable.

Hint: A function g is Riemann integrable if and only if for every $\epsilon > 0$, there exist step functions ϕ and ψ such that $\phi \leq g \leq \psi$ and $\int_a^b (\psi - \phi) < \epsilon$.